A Proposal for the Empirical Determination of the Characteristic Function of a Game

Ernest W. Adams and R. Duncan Luce

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1. Introduction

It is well known that von Neumann and Morgenstern [4] proposed that the study of n-person games be reduced to the study of real-valued set functions v, called characteristic functions, which satisfy

i. $v(\phi) = 0$,

and ii. if R and S are disjoint subsets of I_n, the set of n players, then

$$v(R \cup S) \geq v(R) + v(S).$$

Intuitively, the value v(R) represents numerically what we may term the "strength" of the coalition R. The first condition may be considered as stating that the null set shall not be of any strategic consequence, and the second, that a coalition formed from disjoint sets R and S of players can do everything that R and S can do separately, and possibly more.

Two games v and v' on the set of players I_n are called S-equivalent if there exists a positive constant c and constants a_i such that

$$v(R) = cv'(R) + \sum_{i \in R} a_i.$$

It is argued that two S-equivalent games are subject to the same strategic considerations since c changes only the scale, and the constants a_i represent amounts paid independently of the outcome of the game and may, in fact, be paid before the game ever begins. All theories based on the characteristic function are invariant under S-equivalence.

One might imagine that theories based on such a structure, concerned as they must be with the potential of various coalitions pitted one against another and with the way in which threats of coalition change finally determine the payoffs to the players, would be of considerable significance in both economics and sociology; yet this has not been the case. There appear to be at least two major reasons, aside from the possibility that the characteristic function representation of conflict-of-interest situations may be too simple to cope adequately with most of them. First, none of the published theories (von Neumann and Morgenstern's solutions [4], Shapley's value [5], and Milnor's reasonable outcomes [3]) purports to be a descriptive theory. One of the present authors has put forward a theory which attempts to be more descriptive [2]; but undoubtedly it will have to be modified before it is truly so. In any case, it would be quite impossible at present to determine empirically whether this, or any other theory, is an adequate descriptive theory, for with the exception of contrived experimental situations (see for example [1]) it is not possible practically to ascertain the characteristic function of an existing situation. Thus, a second reason that n-person game theory has not been applied is that the only known way to determine the characteristic function of a game is to obtain the normal form of the game and then to make elaborate calculations using the minimax theorem. Not only is it next to impossible to find the normal form of a game in an existing situation but, considering the billions of strategies that are available in any reasonably complex situation, the minimax theorem calculations would be out of the question.

A mathematician cannot but have faith that ultimately the first difficulty will fall before ingenuity, but it is not so clear that the second - the empirical - difficulty can be overcome. One appears to be in that peculiar but not uncommon situation where the problem is solved in principle, but where it cannot be handled practically because the mathematical term "finite" does not necessarily meet the computer's prayer that it be small. The purpose of this note is to point out that possibly an approximate solution to this problem can be obtained by using a method no more complex than those used to determine an approximation to the von Neumann and Morgenstern numerical utility of a set of alternatives and, while this is empirically difficult, it is vastly simpler than determining the normal form of a game and computing from it the characteristic function.

2. The Proposal

Our idea is very simple: a person is required to report his preferences between pairs of possible coalitions of players, these preferences to be based on his conception of their relative strengths. We shall discuss this more fully later. There is no assumption made that he knows the underlying normal form of the game or the game theory analysis of it; he only states his subjective evaluations of coalition strength - the evaluations which presumably govern his behavior. If these evaluations satisfy the von Neumann and Morgenstern axioms [4] and one other plausible axiom, then we show that there is a set function which is closely related to the utility function determined by the von Neumann and Morgenstern axioms and which satisfies the two conditions of a characteristic function. It is argued that this may reasonably be taken as the characteristic function which the player assumes the game to have and according to which he acts. It is intuitively clear that each of the players may yield a different characteristic function; this will be discussed in §3 .

Let A be a set of alternatives among which a person is to choose. The set A may be extended in the following way: If $R, S \in A$ and if $0 < \alpha < 1$, then $\langle \alpha R, (1-\alpha)S \rangle$ is the prospect "alternative R with probability α and alternative S with probability $1-\alpha$." Let K be the closure of A under the operation $\langle \alpha R, (1-\alpha)S \rangle$. Let " $\prec$ " be a relation defined over K. (The relation $\prec$ will be interpreted to mean that if $R, S \in K$, then "$R \prec S$" if and only if the person imposing $\prec$ on K prefers alternative S to alternative R. If R and S are indifferent we write $R \sim S$, and $R \preceq S$ denotes that S is preferred or indifferent to R.) If $\prec$ is a simple ordering of K which satisfies the von Neumann and Morgenstern axioms, namely,

1. if $R \prec S$, then $R \prec \langle \alpha R, (1-\alpha)S \rangle$
2. if $R \succ S$, then $R \succ \langle \alpha R, (1-\alpha)S \rangle$
3. if $R \prec T \prec S$, then there exists an α such that $\langle \alpha R, (1-\alpha)S \rangle \prec T$,
4. if $R \succ T \succ S$, then there exists an α such that $\langle \alpha R, (1-\alpha)S \rangle \succ T$,
5. $\langle \alpha R, (1-\alpha)S \rangle \sim \langle (1-\alpha)S, \alpha R \rangle$
6. $\langle \beta \langle \alpha R, (1-\alpha)S \rangle, (1-\beta)S \rangle \sim \langle \alpha\beta R, (1-\alpha\beta)S \rangle$
7. If $R \sim S$, then $\langle \alpha R, (1-\alpha)T \rangle \sim \langle \alpha S, (1-\alpha)T \rangle$

then there exists a family $U(\prec)$ of real-valued functions defined over K, called utility functions, such that for each $u \in U(\prec)$, R and $S \in K$, and $0 < \alpha < 1$, the following conditions are satisfied:

i. $R \prec S$ if and only if $u(R) < u(S)$,

and ii. $u(\langle \alpha R, (1-\alpha)S \rangle) = \alpha u(R) + (1-\alpha)u(S)$.

It can also be shown that if $u, u' \in U(\prec)$ then u and u' are linearly related, i.e., $\prec$ determines the utility function up to a linear transformation.

The problem now is to find a method to determine $\prec$ and to give a plausible definition of the characteristic function in terms of the utility functions determined by $\prec$. To do this we shall let A be the set of all subsets of the set of players I_n. Then we have two closely related proposals as to how $\prec$ may be determined over K, now the domain of risk situations involving coalitions.

<u>Proposal 1.</u>

An observer, possibly one of the players of the game, is required to report his preferences in each possible pair of risk situations under the following assumptions:

i. if he chooses a coalition R then he will receive the total payment that R obtains from the situation in which -R forms a coalition and the game is played between R and -R; the alternative ϕ, the empty set, is taken to mean non-participation, i.e., he will neither win nor lose by the choice;

ii. if he chooses $\langle \alpha R, (1-\alpha)S \rangle \in K$, then with probability α he is to receive the payment expected from a choice of coalition R, and with probability $1-\alpha$ he is to receive the payment expected from a choice of coalition S.

Let $\prec$ denote the preference relation so induced on K.

Intuitively, it does not seem unreasonable to suppose that a consistent evaluation of coalition strength should cause $\prec$ to satisfy each of the von Neumann-Morgenstern axioms. While it is unreasonable to expect that people will actually be so consistent, one may hope that in some cases they will be approximately consistent, in other words, that our model of a player's subjective evaluation of coalition strength is approximately correct.

If R and S are two non-overlapping coalitions in A, then the expected payment to $R \cup S$ is at least as much as the sum of the payments to R and S separately. Thus the alternative of receiving the proceeds of $R \cup S$ with a probability of $\frac{1}{2}$ and not participating with a probability of $\frac{1}{2}$ should be no less appealing than the alternative of receiving the proceeds of the coalition R with a probability of $\frac{1}{2}$ and receiving that of S with a probability of $\frac{1}{2}$, provided the variance of the payment is not relevant. Our second proposal does not suffer from such a variance effect. If this intuition is correct, then we may assume the further axiom

8. If $R, S \in A$ and $R \cap S = \phi$, then

$$\left\langle \tfrac{1}{2}R, \tfrac{1}{2}S \right\rangle \precsim \left\langle \tfrac{1}{2}(R \cup S), \tfrac{1}{2}\phi \right\rangle$$

The assumption that the preference relation $\prec$ satisfies the von Neumann and Morgenstern axioms implies the existence of the set $U(\prec)$ of utility functions. It follows immediately from axiom 8 and the properties of the utility function that for any $u \in U(\prec)$, if $R, S \in A$ and $R \cap S = \phi$, then

$$u(R) + u(S) \leqslant u(R \cup S) + u(\phi).$$

Suppose now, that we measure the observer's evaluation of the coalition in terms of deviations from not participating in the game at all, i.e., for $u \in U(\prec)$, define

$$v(R) = u(R) - u(\phi).$$

It is easy to use the above result about u to show that v is a characteristic function. Of course, if $u, u' \in U(\prec)$, then u and u' are linearly related, and so the corresponding v and v' differ only by a change of scale; hence they are S-equivalent. It is actually convenient to enlarge this class. Define C(u) to be the set of all set functions

$$v(R) = c\left[u(R) - u(\phi)\right] + \sum_{i \in R} a_i,$$

where c is a positive constant and the a_i's are constants.

Several theorems are easily proved:

i. if $v \in C(u)$, then v is a characteristic function;

ii. if $v \in C(u)$, then $v' \in C(u)$ if and only if v' is S-equivalent to v;

iii. if $u, u' \in U(\prec)$, then $C(u) = C(u')$.

In addition to the intuitive considerations which suggest that the $v \in C(u)$ with $c = 1$ and $a_i = 0$, i.e., $v(R) = u(R) - u(0)$, is a suitable measure of the observer's evaluation of coalition strength, we can show this is the case if he knows the game structure of the situation and if he bases his evaluations on that knowledge. Specifically, suppose the game is known in normal form and the characteristic function v is determined by the method given by von Neumann and Morgenstern [4]. Let v be extended from A to $\bar{A}$ by the following definition

$$v(\langle \alpha R, (1-\alpha)S \rangle) = \alpha v(R) + (1-\alpha)v(S).$$

Now suppose the observer determines his preference relation according to the following (rational) rule

$$R \prec S \text{ if and only if } v(R) < v(S).$$

It is not difficult to show that $\prec$ satisfies the von Neumann and Morgenstern axioms and axiom 8, and hence a class of utility functions $U(\prec)$ is determined and so a class $C(u)$, $u \in U(\prec)$, of S-equivalent characteristic functions is also determined. One can readily show that $v \in C(u)$. Thus, if a player evaluates the situation according to the von Neumann and Morgenstern theory, our proposed procedure will determine the class of characteristic functions S-equivalent to v; and if all the players so evaluate the situation, this will be reflected in the fact that all empirically determined characteristic functions will be S-equivalent.

<u>Proposal 2.</u>

As before, an observer is required to report his preference in each possible pair of risk situations, but now under the assumptions:

i. if he chooses coalition R, he can expect to receive the average value of payments to players in R, where the game is played between R and $-R$; the alternative ϕ is taken to mean non-participation;

ii. same as in proposal 1;

Let $\prec$ be the preference relation so induced, which in general will not be the same as the relation obtained by the assumptions of proposal 1.

Again it is plausible to assume a consistent evaluation of coalition strength will cause axioms 1 through 7 to be met and, on the basis that $R \cup S$, for disjoint R and S, is stronger than R or S separately, it is reasonable to assume

9. if $R, S \in A$ and $R \cap S = \phi$, then

$$(R \cup S) \succeq \left\langle \frac{|R|}{|R \cup S|} R, \frac{|S|}{|R \cup S|} S \right\rangle$$

where $|R|$ denotes the number of elements in R. From axiom 9 it follows immediately that

if $R, S \in A$ and $R \cap S = \phi$, then

$$u(R \cup S) \geq \frac{|R|}{|R \cup S|} u(R) + \frac{|S|}{|R \cup S|} u(S),$$

where $u \in U(\prec)$.

In this case we define a class $D(u)$, $u \in U(\prec)$, to consist of all set functions

$$v(R) = c\,|R| \left[u(R) - u(\phi)\right] + \sum_{i \in R} a_i,$$

where c is a positive constant and the a_i's are constants. In the case $c = 1$ and $a_i = 0$, $v(R) = |R| \left[u(R) - (\phi)\right]$ is simply the sum of the utility increases to players in R. Essentially the same theorems hold as in proposal 1, namely:

i. if $v \in D(u)$, then v is a characteristic function;

ii. if $v \in D(u)$, then $v' \in D(u)$
if and only if v' is S-equivalent to v;

iii. if $u, u' \in U(\prec)$, then $D(u) = D(u')$;

iv. suppose a game is given whose characteristic function v is extended to K as in proposal 1, and suppose we define $\prec$ so that $R \prec S$

if and only if $\frac{v(R)}{|R|} < \frac{v(S)}{|S|}$, then $\prec$ satisfies axioms 1-7 and 9. It follows that $U(\prec)$ is determined, and $v \in D(u)$, where $u \in U(\prec)$.

8. Discussion

There is little hope that judgments of coalition strength by an observer will in fact satisfy either set of eight axioms, just as the first seven are not generally strictly satisfied by a person's commodity preferences. The pertinent question is whether the model holds approximately and whether the results so obtained, coupled with a descriptive theory based on characteristic functions, lead to suitable predictions.

We can think of two other points which will surely be raised and which it seems appropriate to discuss. It may be objected that our determination of the characteristic function is entirely subjective and that in all likelihood it would have little relation to the characteristic function determined from the normal form of the game, were we able to calculate it. In the present presentations of game theory the subjective factors enter when the players determine a utility function over the set of possible outcomes, and from these a unique characteristic function is determined. It is not at all obvious, even for a person aware of the utility functions over the possible outcomes, that he will behave in accordance with the von Neumann-Morgenstern theory. He may react to his evaluations of coalition alternatives more or less independently of his evaluations of the outcomes of the game in normal form. But if this is the case, then it is a player's subjective characteristic function, and not the objective ones of the game, which actually determines his behavior, and so it will be needed for predictions of his behavior.

The last point may be relevant to account for one of the outstanding results of a game experiment run at RAND [1]. Two different 4-person constant-sum games were each presented to subjects in what amounted to a 0,1 reduced form and in an S-equivalent form. In both cases the data from the two S-equivalent games were strikingly different, though theoretically both games require exactly the same strategic considerations. It appears from these data that the subjects dealt reasonably adequately with the strategic features when the games were in the 0,1 reduced form, but that certain superficial aspects of the mode of presentation led them to a false evaluation of coalition strength when the games were presented in the S-equivalent forms. That is to say, probably the subjects were not in fact responding to the given characteristic function at all, but to a subjective one which they "derived" from the given one.

A second objection which can be raised, and one which is of vital importance, is that there is little reason to hope that two different observers of the same situation will have evaluations of coalition strengths which lead to the same or S-equivalent characteristic functions. Should two different evaluations arise, none of the current theories will be applicable since they deal only with a unique characteristic function. It is not at all certain that this is a criticism of the present proposal rather than an observation that game theory, as a descriptive theory, may be over-simple, at least with regard to the current assumption that all players respond to the same characteristic function derived from their different evaluations of possible outcomes.

These remarks suggest that there may be a need for two theoretical developments. First, assuming that each player of a game bases his actions in the situation on his own characteristic function, an equilibrium theory describing the payments and coalitions which may be expected should be developed. Second, the notion of the normal form of a game should be modified in such a way that one can derive from it a distinct characteristic function for each player. Such a theory probably should include as a special case the von Neumann and Morgenstern reduction of the normal form to a single characteristic function. One simple possibility is to assume that each of the players has his own utility function over the possible outcomes and that he has beliefs as to the utility functions of the other players, beliefs which will in general be in error. This assumption results in an objective normalized game and for each of the players a fictional game which is the one he believes to exist. Assuming that each player responds only to his beliefs, there is associated with him the characteristic function of the fictional game. If each of the fictional games is identical to the objective game, then the theory reduces to the von Neumann and Morgenstern one.

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